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REALIZATIONS OF HIGHER-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract We discuss the realization problem for nonlinear systems which are not given by a nonlinear input-output map, but as a set of smooth higher-order differential equations involving the inputs and outputs. We show that under general conditions the existence of realizations involving auxiliary driving variables is guaranteed, but that for input-output realizations extra integrability conditions have to be imposed. The realization procedure uses the concept of zero dynamics which is briefly discussed.

1. Introduction

Nonlinear realization theory has mainly concentrated so far on the problem of realizing a nonlinear input-output map

$$y(t) = F[u(\tau);\ 0\le\tau\le t],\ t\ge 0,\ u\in R^m,\ y\in R^p \qquad (1.1)$$

(sometimes given in Volterra series, or generating power series form) as a (minimal) input-state-output system

$$\begin{aligned} \dot{x} &= f(x,u) \qquad x(0) = x_0 \in M \\ y &= h(x,u) \end{aligned} \qquad (1.2)$$

living on an n-dimensional state space manifold M. On the other hand many (physical) nonlinear systems have a natural description not in terms of an input-output map but as a set of nonlinear differential equations involving the inputs and outputs and their time-derivatives up to an arbitrary (finite) order k

$$R_i(y,\dot{y},\dots,y^{(k)},u,\dot{u},\dots,u^{(k)})=0 \quad i = 1,\dots,\ell \qquad (1.3)$$

e.g. nonlinear electrical networks and mechanical systems. As a matter of fact, as was argued by Willems [17], in many cases the natural starting point for the description of differentiable systems is a set of higher-order differential equations

$$P_i(w,\dot{w},\dots,w^{(k)},\xi,\dot{\xi},\dots,\xi^{(k)})=0, \quad i = 1,\dots,\ell \quad w\in R^q,\ \xi\in R^s \qquad (1.4)$$

involving a set of **external** variables w (roughly speaking the inputs and outputs) and a set of (auxiliary) **internal** variables ξ. For example, in the case of an electrical network the components of w denote the voltages and currents at the external ports, while ξ will denote some internal variables like voltages and/or currents of some elements inside the network. Notice that input-state-output systems (1.2) are also of the form (1.4), with w consisting of y and u, and ξ equal to the state variables x. Furthermore, the **interconnection** of systems (1.2) results in a system of this form.
The **input-output realization problem** for an **external system** (1.4) can be stated as follows [17,12]. Construct a state space manifold M, a permutation matrix T: $R^q \to R^q$ and mappings $f(x,u)$, $h(x,u)$ such that the **external behavior** of the input-state-output system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \qquad x\in M \qquad (1.5)$$

with $Tw = \mathrm{col}(y,u)$, $y\in R^p$, $u\in R^m$, $m+p = q$ **equals** the **external behavior** of the external system (1.4). The external behaviors of (1.4) and (1.5) are defined in the following way. We restrict ourselves entirely to **smooth** external behaviors. The smooth external behavior of (1.4) equals

$$\{w(\cdot): R\to R^q \mid \exists \text{ smooth mapping } \xi(\cdot): R\to R^s \text{ such that } w(t),\ \xi(t) \text{ satisfies (1.4) for all } t\} \qquad (1.6)$$

and the smooth external behavior of (1.5) is

$$\{w(\cdot): R\to R^q,\ Tw = \mathrm{col}(y,u) \mid \exists \text{ smooth trajectory } x(\cdot): R\to M \text{ such that } w(t),\ x(t) \text{ satisfies (1.5) for all } t\} \qquad (1.7)$$

For technical reasons we shall assume throughout that all our systems are **complete**, i.e., any locally defined solution w(t) of (1.4) can be extended to a solution for all $t\in R$ for any smooth mapping $\xi(\cdot)$. Similarly the vectorfields $f(x,u(t))$ in (1.5) are assumed to be complete for any smooth $u(\cdot)$.
With respect to the smoothness of the functions w(t) we remark that once we have obtained an input-output realization (1.5) of (1.4), then the external behavior of the realization can be **enlarged** by allowing for a broader class of input functions u(t) (for example piecewise continuous). Then also the external behavior of (1.4) can be enlarged by **defining** it to be equal to this enlarged external behavior of (1.5).
Notice that in our realization problem the initial state x_0, as in (1.2), does not play any role. Furthermore the inputs and outputs need not be a priori given; the realization procedure has to decide which part of the w-vector can be correctly called inputs and which remaining part outputs. (Note that instead of permutation matrices we may also allow for more general transformations on the external variables.) Also autonomous systems (i.e. without inputs) can be dealt with in this way.
For **linear** systems a similar point of view was already advocated by Rosenbrock [11] by considering equations of the form

$$\begin{aligned} T(\tfrac{d}{dt})\xi &= U(\tfrac{d}{dt})u \\ y &= V(\tfrac{d}{dt})\xi + W(\tfrac{d}{dt})u \end{aligned} \qquad (1.8)$$

where T, U, V and W are linear differential operators (see also Blomberg [1], Wolovich [18]). While one can argue that in the linear case the difference with the usual transfer matrix approach is mainly on the conceptual level, in the nonlinear case the input-output map point of view and the higher-order differential equations approach are really **not** equivalent. This was already argued in [12]. The present paper is largely based on [15], to which we refer for more background and details. This last paper in turn was much inspired by a paper of Schumacher [16] where the realization problem in the linear case was treated (see also [17]). Finally, we mention the connection with some recent work of Fliess dealing with systems of the form (1.3) [5].

2. A realization procedure

Consider the external nonlinear system

$$P_i(w,\dot w,\ldots,w^{(k)},\xi,\dot\xi,\ldots,\xi^{(k)}) = 0 \qquad (2.1)$$
$$i = 1,\ldots,\ell, \qquad w \in R^q,\ \xi \in R^s$$

where the P_i are **smooth** functions. Define $z = \mathrm{col}(w,\xi) \in R^q \times R^s$. We can associate with (2.1) the system with linear dynamics

$$\frac{d}{dt}\begin{bmatrix} z \\ \dot z \\ \vdots \\ z^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I_{q+s} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & I_{q+s} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}\begin{bmatrix} z \\ \dot z \\ \vdots \\ z^{(k)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I^{q+s} \end{bmatrix}\bar v \qquad (2.2a)$$

and nonlinear output functions

$$P_i(z,\dot z,\ldots,z^{(k)}) \qquad i = 1,\ldots,\ell \qquad (2.2b)$$

Here $\bar v \in R^{q+s}$ denote arbitrary (smooth) functions called the **driving variables** of the system. We note that the **external behavior** of (2.1) is precisely given as the first q components of all time-functions z(t) generated by the dynamics (2.2a) with the property that the outputs $P_i(z,\dot z,\ldots,z^{(k)})$ are identically zero. In geometric terms this means that we have to compute the "maximal controlled invariant submanifold" $N^* \subset R^{(k+1)(q+s)}$ contained in the set $P_1 = \ldots = P_\ell = 0$. (A submanifold $N \subset M$ of a control system with state space M is controlled invariant if it can be made invariant by applying feedback to the system, cf. section 3.)

Theorem 1 [15] For any $i = 1,\ldots,\ell$, define the characteristic number ρ_i as the smallest nonnegative integer such that for some $j \in \{1,\ldots,q+s\}$ and some $(z,\dot z,\ldots,z^{(k)})$

$$\frac{\partial P_i}{\partial z_j^{(k-\rho_i)}}(z,\dot z,\ldots,z^{(k)}) \neq 0 \qquad (2.3)$$

If ρ_i is not defined (i.e., if P_i is a constant function), then we set $\rho_i = k$. Furthermore define the $\ell \times (q+s)$ matrix

$$A(z,\dot z,\ldots,z^{(k)}) = \left(\frac{\partial P_i}{\partial z_r^{(k-\rho_i)}}(z,\dot z,\ldots,z^{(k)})\right)_{\substack{i=1,\ldots,\ell \\ r=1,\ldots,q+s}} \qquad (2.4)$$

Assume that rank $A(z,\dot z,\ldots,z^{(k)}) = \ell$ for any $(z,\dot z,\ldots,z^{(k)})$ gained in the set $P_1 = \ldots = P_\ell = 0$. Then the maximal controlled invariant submanifold contained in $P_1 = \ldots = P_\ell$ exists and is given as

$$N^* = \{(z,\dot z,\ldots,z^{(k)}) \mid \frac{d^r P_i}{dt^r}(z,\dot z,\ldots,z^{(k)}) = 0,\ r=0,1,\ldots,\rho_i,\ i=1,\ldots,\ell\} \qquad (2.5)$$

Denote $B(z,\dot z,\ldots,z^{(k)}) = \mathrm{col}\left(\frac{d^{\rho_1+1}}{dt^{\rho_1+1}}P_1,\ldots,\frac{d^{\rho_\ell+1}}{dt^{\rho_\ell+1}}P_\ell\right)$ and $x = (z,\dot z,\ldots,z^{(k)})$. Then the feedback which makes N^* invariant is given by $\bar v = \alpha(x)+\beta(x)v$, with $v \in R^m$ and $m = q+s-\ell$, where $\alpha(x)$ is a solution of

$$A(x)\alpha(x) + B(x) = 0 \qquad (2.6)$$

and $\beta(x)$ is a $(q+s)\times m$ matrix of full rank satisfying

$$A(x)\beta(x) = 0 \qquad (2.7)$$

After applying such feedback $\bar v = \alpha(x) + \beta(x)v$ to (2.2a) the resulting system is tangent to N^* and hence we obtain an affine control system **defined** on N^*

$$\dot x = g_0(x) + \sum_{j=1}^m v_j g_j(x),\ x = (z,\ldots,z^{(k)}) \in N^* \qquad (2.8a)$$

Furthermore since N^* is a submanifold of $R^{(k+1)(q+s)}$ the projection of N^* onto the first q components of $R^{(k+1)(q+s)}$ is a smooth mapping

$$w_j = G_j(x),\quad j = 1,\ldots,q,\ x \in N^* \qquad (2.8b)$$

System (2.8) obtained in Theorem 1 is a **driven realization** of the external system (2.1); this means that the totality of functions w(t) generated by (2.8) by considering different (smooth) driving functions $v(\cdot)$ and initial states in N^* coincides with the external behavior of (2.1). Driven state space systems (2.8) were treated in [17,12].
The second step of our realization procedure is to produce an input-output realization from the driven realization (2.8). Roughly speaking we have to decide which components of the w-vector can serve as inputs. This is done by maximally reducing the number of integrations from the driving variables v to the outputs w in (2.8). For this we apply the algorithm to compute the minimal **conditioned invariant** distribution S^* containing the input distribution span $\{g_1,\ldots,g_m\}$. However, to obtain an input-state-output system with the **same** external behavior as (2.1), we have to impose extra integrability conditions on the distributions involved in every step of the algorithm. These conditions generalize the conditions in [6] for realizations of external systems of the form

$$y_i - a_i(y,u,\dot u) = 0 \quad i = 1,\ldots,p \qquad (2.9)$$

and we refer to [15].
In conclusion, if the assumption of Theorem 1 is met then there exists a **driven realization** of (2.1). In general there does **not** exist an input-output realization; extra structural conditions have to be added. We note that for **linear** systems these conditions are trivially satisfied, and so indeed linear external systems always admit input-output realizations ([16,17]). Finally, it can be shown [15] how driven, respectively input-output realizations can be reduced, under regularity assumptions, to **minimal** driven, respectively input-output realizations.
Let us now return to the full rank assumption of the A-matrix in Theorem 1. First let us consider a **linear** external system

$$P(\tfrac{d}{dt})\, z(t) = 0 \qquad z = \mathrm{col}(w,\xi) \tag{2.10}$$

where $P(s)$ is an $\ell \times (q+s)$ polynomial matrix. It can be easily seen that (2.10) satisfies the assumption of Theorem 1 if and only if $P(s)$ is a **row proper** matrix [18]. Now let us premultiply $P(s)$ by a unimodular matrix $U(s)$ (i.e., $\det U(s)$ is a nonzero constant). It is clear that the external behaviors of (2.10) and that of $\bar{P}(\frac{d}{dt})z(t) = 0$, with $\bar{P}(s) = U(s)P(s)$ are the **same**. On the other hand it is well-known that for every $\ell \times (q+s)$ polynomial $P(s)$ there exists an $\ell \times \ell$ unimodular matrix $U(s)$ such that

$$U(s)P(s) = \left(\frac{P'(s)}{0}\right) \tag{2.11}$$

where $P'(s)$ is an $\ell' \times (q+s)$ row proper matrix ($\ell' \leq \ell$). Clearly the external behaviors of $P(\frac{d}{dt})z(t) = 0$ and $P'(\frac{d}{dt})z(t) = 0$ are equal. Therefore in the linear case the assumption of Theorem 1 is satisfied **without loss of generality.** We shall show that this observation more or less also extends to the nonlinear case.

If a time function $z(t) = (w(t),\xi(t))$ satisfies for a certain j, $P_j(z(t),\dot{z}(t),\dots,z^{(k)}(t)) = 0$, then $z(t)$ also satisfies for any integer $\nu_j \geq 0$

$$P_j^{(\nu_j)}(z,\dot{z},\dots,z^{(k)}) = 0 \tag{2.12}$$

where $P_j^{(\nu_j)} = \dfrac{d^{\nu_j}P_j}{dt^{\nu_j}}$. Now consider smooth functions $\phi\colon U \subset \mathbb{R}^\ell \to \mathbb{R}$, with U an open neighbourhood of $0 \in \mathbb{R}^\ell$, satisfying

$$\phi(0,0,\dots,0) = 0 \tag{2.13a}$$

$$\phi(\cdot,0,\dots,0)\colon \mathbb{R} \to \mathbb{R} \text{ is a local diffeomorphism around } 0 \in \mathbb{R}. \tag{2.13b}$$

Let $s \in \{1,\dots,\ell\}$. Replace the set of eqations (2.1) by

$$P_i(z,\dot{z},\dots,z^{(k)}) = 0 \qquad i = 1,\dots,\ell,\ i \neq s \tag{2.14a}$$

$$\phi(P_s,P_1^{(\nu_1)},\dots,P_{s-1}^{(\nu_{s-1})}, P_{s+1}^{(\nu_{s+1})},\dots,P_\ell^{(\nu_\ell)})\,(z,\dots,z^{(k)}) = 0 \tag{2.14b}$$

It immediately follows from (2.12) and (2.13) that the external behavior defined by (2.14) equals the external behavior of (2.1). Hence any ϕ satisfying (2.13) defines an equivalence or **unimodular** transformation on external systems (2.1).

Now consider an external system (2.1) for which the rank of $A(z,\dot{z},\dots,z^{(k)})$ is less than ℓ. **Assume** it to be constant. Denote the rows of A by $A_i(z,\dot{z},\dots,z^{(k)})$. It follows that there exist non-trivial functions $\alpha_i(z,\dot{z},\dots,z^{(k)})$ such that

$$\sum_{i=1}^{\ell} \alpha_i(z,\dots,z^{(k)})A_i(z,\dots,z^{(k)}) = 0 \tag{2.15}$$

Let ρ_j be the smallest integer for which the function α_j in (2.15) is not identically zero. **Assume** that this function α_j is nowhere zero. Then there exist functions $\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_\ell$ such that

$$A_j = \lambda_1A_1 + \dots + \lambda_{j-1}A_{j-1} + \lambda_{j+1}A_{j+1} + \dots + \lambda_\ell A_\ell \tag{2.16}$$

It can be proved [15] that for $i = 1,\dots,\ell$

$$A_i = \left(\frac{\partial}{\partial z^{(k-\rho_j)}}P_i^{(\rho_i-\rho_j)},\dots,\frac{\partial}{\partial z_{q+s}^{(k-\rho_j)}}P_i^{(\rho_i-\rho_j)}\right) \tag{2.17}$$

Now consider the function P_j together with the $\ell-1$ functions $P_i^{(\rho_i-\rho_j)}$ $i = 1,\dots,j-1,j+1,\dots,\ell$, as functions of $(z_1^{(k-\rho_j)},\dots,z_{q+s}^{(k-\rho_j)})$. It follows from (2.16) and (2.17) that

$$dP_j = \lambda_1dP_1 + \dots + \lambda_{j-1}dP_{j-1} + \lambda_{j+1}dP_{j+1} + \dots + \lambda_\ell dP_\ell \tag{2.18}$$

where d means differentiation to $z^{(k-\rho_j)}$. This implies ([15, Lemma 6.2]) that there exists a function ϕ satisfying (2.13) such that

$$\frac{\partial}{\partial z_r^{(k-\rho_j)}}(\phi\circ(P_j,P_1,\dots,P_{j-1},P_{j+1},\dots,P_\ell)) = 0 \qquad r = 1,\dots,q+s \tag{2.19}$$

Hence for the external system obtained after the application of the unimodular transformation defined by ϕ the j-th characteristic number is strictly greater than ρ_j (while the others remain the same). Consequently the j-th row of the A-matrix is different. If this new A-matrix has rank ℓ then we stop. Otherwise we apply the same procedure again. Since in every step one characteristic number increases and the characteristic numbers are bounded by k there are only two possibilities:

1° After a finite number of steps we obtain an A-matrix with full rank.

2° After a finite number of steps we obtain an external system consisting of ℓ' ($\ell' \leq \ell$) equations with an A-matrix of rank ℓ' together with $\ell-\ell'$ equations of the form constant = 0.

In case 1° we are done. In case 2° it depends on the constant equations. If all constants are zero then we are also done, while if some constants are **not** zero the set of equations is inconsistent and N^* does not exist. In concluse we have shown, under regularity assumptions, that an external system for which N^* exists satisfies without loss of generality the assumption of Theorem 1.

3. An aside zero dynamics

Recently there has been considerable interest in generalizing the notion of transmission zeros for linear systems to the nonlinear case ([2,3,4,10,13,14]). These attempts are based on the following **geometric** interpretation of the zeros of a transfer matrix $G(s)$. Take any minimal realization (A,B,C) of $G(s)$ and compute V^* and R^* (the maximal controlled invariant, respectively controllability subspace contained in Ker C). For any feedback $u = Fx$ such that $(A + BF)V^* \subset V^*$ we obtain an induced mapping $\bar{A}\colon V^*/R^* \to V^*/R^*$ which moreover does **not** depend on F. The spectrum of $\bar{A}$ are exactly the transmission zeros of $G(s)$. Hence the dynamics $\dot{z} = \bar{A}z$, with $z \in V^*/R^*$, can be correctly called the **zero dynamics.**

To generalize this to the nonlinear case we note that the concept of V^* has at least two different interpretations, which in the **linear** case coincide. First interpretation is to consider V^* as the maximal subspace contained in Ker C in which one can stay by suitably choosing the input u. In the second interpretation one considers the **foliation** $x + V^*$, $x \in \mathbb{R}^n$, and the flow of $\dot{x} = (A + BF)x$, where $(A + BF)V^* \subset V^*$, leaves this foliation invariant. As noted in [7,9] this second

interpretation is the right one for problems like disturbance and input-output decoupling, and can be generalized to the nonlinear case by using (integrable) distributions. Under regularity assumptions one can compute the maximal controlled invariant distribution Δ^* contained in the kernel of the differentials of the output functions. In [2,3,4,10] it is shown that in the single-input case (where "$R^* = 0$") this leads to a satisfying definition of zero dynamics.
On the other hand in [13,14] the first interpretation of V^* was stressed in order to defined the zero dynamics. Les us now extend the observations made in [13,14]. Consider a minimal nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \quad u\in R^m, \quad y\in R^p, \ x\in M \\ y &= h(x) \end{aligned} \tag{3.1}$$

(The case $y = h(x,u)$ can be similarly treated.) Clearly the **free dynamics** of (3.1) are the dynamics for $u = 0$. This is the generalization of the linear concept of poles. In the same way the **zero dynamics** (or **clamped dynamics**, cf.[13]) can be defined as the dynamics which are compatible with the constraints $y = 0$. (Since $y = 0$ may not have an intrinsic meaning it could be replaced by $y = c$, with c a constant vector. Note that also $u = 0$ may not have an intrinsic meaning.)

Definition 2 [13,14] A submanifold $N \subset M$ is called controlled invariant for (3.1) if there exists a feedback $u = \alpha(x)$ such that the vectorfield $\dot{x} = f(x,\alpha(x))$ is **tangent** to N (and so its solutions remain in N).

The maximal controlled invariant submanifold contained in the set $h(x) = 0$ will be denoted as N^*. However, N^* may not alway be defined (for example if two output functions satisfy $h_1(x) = h_2(x)+1$). Sufficient conditions for the existence of N^* are given in the following extension of Theorem 1. Define for any $i = 1,\dots,p$, ρ_i as the smallest nonnegative integer such that for some $k \in \{1,\dots,m\}$ and some (x,u)

$$\frac{\partial}{\partial u_k}(L_f^{\rho_i+1} h_i)(x,u) \neq 0 \tag{3.2}$$

(Here $L_f h$ denotes the Lie-derivative of $h(x,u)$ along $\dot{x} = f(x,u)$ with u treated as a parameter.) If $\rho_i < \infty$, $i = 1,\dots,p$, we can define the $p\times m$ matrix

$$A(x) = \left(\frac{\partial}{\partial u_j}(L_f^{\rho_i+1} h_i)(x,u)\right)_{\substack{i=1,\dots,p \\ j=1,\dots,m}} \tag{3.3}$$

Theorem 3 Assume that $\rho_i < \infty$, $i = 1,\dots,p$, and that the matrix $A(x)$ has rank p. Then the functions $h_i, L_f h_i, \dots, L_f^{\rho_i} h_i$, $i = 1,\dots,p$, which by definition of ρ_i only depend on x, are independent. The maximal controlled invariant submanifold $N^* \subset M$ contained in $h_1=\dots=h_p = 0$ exists and is given as

$$N^* = \{x\in M \mid h_i = \dots = L_f^{\rho_i} h_i = 0, \ i=1,\dots,p\} \tag{3.4}$$

The feedback $u_j = \alpha_j(x)$, $j = 1,\dots,m$, which makes N^* invariant is given as a solution of

$$(L_f^{\rho_i} h_i)(x,\alpha_1(x),\dots,\alpha_m(x)) = 0 \quad i=1,\dots,p \tag{3.5}$$

Remark: Theorem 1 can be immediately deduced from Theorem 3 using the special form of the dynamics (2.2a).

Proof The independence of the functions follows as in [8, Lemma 3.10]. The functions $L_f^{\rho_i} h_i$ are by definition of ρ_i affine in u, and so (3.5) can be globally solved for α_j due to the full rank of $A(x)$. Define $\bar{f}(x) = f(x,\alpha(x))$ for a solution α of (3.5). Then for $r = 0,1,\dots,\rho_i-1$

$$L_{\bar{f}}^{r+1} h_i = L_f^{r+1} h_i \quad \text{and} \quad L_{\bar{f}}^{\rho_i+1} h_i = 0$$

Hence N^* defined by (3.4) is controlled invariant. Furthermore, since $h_i,\dots,L_f^{\rho_i} h_i$ do not depend on u, all these functions have to be zero on an arbitrary controlled invariant submanifold contained in $h_1 = \dots = h_p = 0$. Hence N^* is maximal. □

In case $p = m$ the solution $u = \alpha(x)$ of (3.5) is uniquely determined. In case $p \le m$ then if $\bar{\alpha}(x)$ is a solution of (3.5) then so is $\bar{\alpha}(x) + z(x)$ where $z(x)$ satisfies $A(x)z(x) = 0$. Hence ker $A(x)$ defines an m-p dimensional distribution on N^*. Locally we can find vectorfields $g_1,\dots,g_{m-p}$ on N^* such that

$$\ker A(x) = \operatorname{span}\{g_1(x),\dots,g_{m-p}(x)\} \tag{3.6}$$

Restricting the vectorfield $\bar{f}(x) = f(x,\alpha(x))$, which is tangent to N^*, to the vectorfield $g_0(x)$ defined on N^* we then have obtained the affine system

$$\dot{x} = g_0(x) + \sum_{j=1}^{m-p} v_j g_j(x) \tag{3.7}$$

on N^* with inputs v. Denote the strong accessibility distribution of (3.7) by L_0; this will be the nonlinear generalization of R^*. If dim L_0 is constant then (locally) we may factor out N^* by L_0 to a manifold N^*/L_0, and g_0 projects to a vectorfield $\bar{g}_0$ on N^*/L_0. The dynamics

$$\dot{\bar{x}} = \bar{g}_0(\bar{x}), \quad \bar{x} \in N^*/L_0 \tag{3.8}$$

can be called the **zero-dynamics** of (3.1). Note that we can give a similar definition in case the assumptions of Theorem 3 are **not** satisfied, but N^* **does** exist.

In case the assumptions of Theorem 3 are satisfied the maximal controlled invariant distribution Δ^* contained in $dh_1 =\dots= dh_p$ is given as $dh_i =\dots= dL_f^{\rho_i} h_i = 0$, $i=i,\dots,p$ [8]. Hence in this case N^* is an integral manifold of Δ^*. Therefore if we **assume** that N^* contains an **equilibrium point** for $\dot{x} = f(x,0)$ it also follows from the controlled invariance of Δ^* that N^* is controlled invariant, cf. [2,3,4]. However from Theorem 3 it follows that this extra assumption is not needed.
In case the assumptions of Theorem 3 are **not** satisfied the two approaches are really different. If Δ^* and N^* both exist then generally N^* will not be a leaf of Δ^*. In fact, since N^* need not be invariant for the input vectorfields (contrary to Δ^*), the dimension of N^* generally will be greater than the dimension of Δ^*. Finally, in section 2 we have shown that in the special case of systems of the form (2.2) the computation of N^* can be reduced to the case that the A-matrix has full rank. It is an interesting problem to what extent this holds for general nonlinear systems (3.1).

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